LAMINAR BOUNDARY LAYER ON AN INFINITE DISC ROTATING IN A GAS

(LAMINARNYI POGRANICHNYI SLOI NA NEOGRANICHENNOM DISKE VRASHCHAIUSHCHEMSIA V GAZE)

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In the dynamics of viscous incompressible fluids one is familiar with the Karman problem of an infinite disc rotating at constant angular velocity and generating laminar motion in the fluid medium which is immediately adjacent to it. The solution to this problem is one of the examples of exact solutions to the Navier-Stokes equation [1]. The heat transfer problem of the fluid under constant temperature conditions of the disc surface was solved (also exactly) by Millsaps and Pohlhausen [2].

In this article we show how, by solving the appropriate gasdynamic problem, with certain conditions and simplifications, we can find a solution of both these problems.

1. Formulation of the problem. We imagine an infinite plane disc rotating uniformly about an axis perpendicular to its plane in a space filled with viscous gas. Let the rotational axis be z, and the disc plane coincide with that of $z = 0$. Using cylindrical coordinates (r, θ, z) we can write down the basic equations which define motion and heat transfer in a viscous flow of gas (see, for instance, [3]). We will assume that the following conditions apply: the gas is a perfect one; the flow is steady; the flow parameters are independent of the angular coordinate θ; there are no body forces; there is no mass heat flux from outside; the so-called "second viscosity coefficient" differs from the basic one only in respect of a constant multiplier, $\mu_2 = a\mu$, $|a| = O(1)$. The boundary conditions for temperature and velocity vector components in the disc problem will be as follows:

$$\begin{aligned} &T(r, \theta, 0) = T_w, \quad u_r(r, \theta, 0) = 0, \quad u_\theta(r, \theta, 0) = r\omega, \\ &T(r, \theta, \infty) = T_\infty, \quad u_r(r, \theta, \infty) = 0, \quad u_\theta(r, \theta, \infty) = 0, \quad u_z(r, \theta, 0) = 0 \end{aligned} \qquad (1.1)$$

2. Transformation to dimensionless variables and simplification of the equation. We introduce two additional assumptions, namely, we assume the Prandtl number $\sigma = \mu c_p / \lambda$ to be constant and we also assume the following law relating viscosity with temperature:

$$\mu / \mu_\infty = (T / T_\infty)^n$$

Furthermore, we select the scales of our required variables to be quantities whose order of magnitude corresponds to the maximum values of the variables themselves; the scale for axial velocity is chosen by analogy with the solution for the incompressible fluid. Independent variables are also rendered non-dimensional while the scale in the axial direction is of the order or thickness of the hydrodynamic viscous layer in an incompressible fluid. As regards the radial length scale the choice is based on the condition of maintaining a minimum number of dimensionless parameters in the equations.

The dimensionless variables are introduced using these formulas

$$r = \frac{\sqrt{c_p T_\infty}}{\omega}\bar{r}, \quad z = \sqrt{\frac{\nu_\infty}{\omega}}\bar{z}, \quad u_r = \omega r F, \quad u_\theta = \omega r G, \quad u_z = \sqrt{\nu_\infty \omega}\, N \tag{2.1}$$

$$\mu = \mu_\infty Q^n, \quad T = T_\infty Q, \quad \rho = \rho_\infty D, \quad p = \rho_\infty c_p T_\infty P$$

Here F, G, N. Q, D, P, are, in general, functions of r and z. Our equations in dimensionless form will be as follows (bars above r and z have been dropped):

Equation of continuity

$$\frac{1}{r}\frac{\partial (r^2 DF)}{\partial r} + \frac{\partial (DN)}{\partial z} = 0 \tag{2.2}$$

Equations of motion in components in three directions

$$\begin{aligned} D\left[F\frac{\partial (rF)}{\partial r} + N\frac{\partial F}{\partial z} - G^2\right] = \\ = -r^{-1}\frac{\partial P}{\partial r} + \frac{2}{3K}r^{-1}\frac{\partial}{\partial r}\left\{Q^n\left[3\frac{\partial (rF)}{\partial r} - \frac{1}{r}\frac{\partial (r^2F)}{\partial r} - \frac{\partial N}{\partial z}\right]\right\} + \\ + \frac{\partial}{\partial z}\left[Q^n\left(\frac{\partial F}{\partial z} + \frac{1}{K}r^{-1}\frac{\partial N}{\partial r}\right)\right] + \frac{2}{K}Q^n r^{-2}\left[\frac{\partial (rF)}{\partial r} - F\right] \end{aligned} \tag{2.3}$$

$$\begin{aligned} D\left[F\frac{\partial (rG)}{\partial r} + N\frac{\partial G}{\partial z} + FG\right] = \frac{\partial}{\partial z}\left(Q^n\frac{\partial G}{\partial z}\right) + \frac{1}{K}r^{-1}\frac{\partial}{\partial r}\left\{Q^n\left[\frac{\partial (rG)}{\partial r} - G\right]\right\} + \\ + \frac{2}{K}Q^n r^{-2}\left[\frac{\partial (rG)}{\partial r} - G\right] \end{aligned} \tag{2.4}$$

$$\begin{aligned} \frac{1}{K}D\left(rF\frac{\partial N}{\partial r} + N\frac{\partial N}{\partial z}\right) = -\frac{\partial P}{\partial z} + \frac{1}{K}\frac{\partial}{\partial z}\left\{Q^n\left[2\frac{\partial N}{\partial z} - r^{-1}\frac{\partial (r^2F)}{\partial r}\right]\right\} + \\ + \frac{1}{K}r^{-1}\frac{\partial}{\partial r}\left\{Q^n r\left[\frac{\partial (rF)}{\partial z} + \frac{1}{K}\frac{\partial N}{\partial r}\right]\right\} \end{aligned} \tag{2.5}$$

Energy equation

$$D\left(rF\frac{\partial Q}{\partial r}+N\frac{\partial Q}{\partial z}\right)-\left(rF\frac{\partial P}{\partial r}+N\frac{\partial P}{\partial z}\right)=\frac{1}{K\sigma}r^{-1}\frac{\partial}{\partial r}\left(rQ^n\frac{\partial Q}{\partial r}\right)+\frac{1}{\sigma}\frac{\partial}{\partial z}\left(Q^n\frac{\partial Q}{\partial z}\right)+$$
$$+Q^n\left\{\frac{2}{K}\left[\left(\frac{\partial (rF)}{\partial r}\right)^2+F^2+\left(\frac{\partial N}{\partial z}\right)^2\right]+r^2\left(\frac{\partial G}{\partial z}\right)^2+\left(r\frac{\partial F}{\partial z}+\frac{1}{K}\frac{\partial N}{\partial r}\right)^2+\right.$$
$$\left.+\frac{1}{K}\left[\frac{\partial (rG)}{\partial r}-G\right]^2\right\}+\frac{\alpha}{K}Q^n\left[\frac{\partial (rF)}{\partial r}+F+\frac{\partial N}{\partial z}\right]^2 \qquad (2.6)$$

Equation of state

$$P=\frac{\varkappa-1}{\varkappa}DQ \qquad (2.7)$$

In Equations (2.3)-(2.6) there is a dimensionless parameter K, defined by the equation

$$K=c_pT_\infty\nu_\infty^{-1}\omega^{-1} \qquad (2.8)$$

This quantity is the basic parameter of similarity for the given problem and can be regarded as a combination based on the circumferential velocity of the disc, the Mach number and Reynolds number

$$K=\frac{1}{\varkappa-1}R_\infty(r)M_\infty^{-2}(r)$$

A very wide class of flows exists which corresponds to the condition $K \gg 1$.

Equations obtained from Equations (2.2)-(2.7) in the limiting case with $K\to\infty$, will be called boundary layer equations on a rotating disc. In particular, from Equation (2.5) and from the condition of constant pressure at infinity, it follows that over the whole flow

$$P=\text{const}=\frac{\varkappa-1}{\varkappa} \qquad (2.9)$$

The boundary conditions (1.1) in dimensionless form are as follows:

$$\begin{array}{llll} Q(r,\ 0)=Q_w, & F(r,\ 0)=0, & G(r,\ 0)=1, & \\ Q(r,\infty)=1, & F(r,\infty)=0, & G(r,\infty)=0, & N(r,\ 0)=0 \end{array} \qquad (2.10)$$

3. Construction of an exact solution for the boundary layer. To construct a solution to the system of Equations (2.2)-(2.7) for $K\to\infty$ we will carry out a transformation similar to Dorodnitsyn's for a two-dimensional boundary layer in a gas [4]. In fact, instead of r and z we introduce new independent variables

$$\eta=r, \qquad \zeta=\int_0^z D\,dz \qquad (3.1)$$

We also introduce a new unknown quantity H, to replace the function N

$$H=ND+\eta F\frac{\partial\zeta}{\partial r} \qquad (3.2)$$

while the index n in the viscosity temperature relation will be assumed to be unity; Equations (2.2)-(2.4) and (2.6) are brought to the form

$$\frac{1}{\eta}\frac{\partial(\eta^2 F)}{\partial\eta}+\frac{\partial H}{\partial\zeta}=0,\quad F\frac{\partial(\eta F)}{\partial\eta}+H\frac{\partial F}{\partial\zeta}-G^2=\frac{\partial^2 F}{\partial\zeta^2},\quad F\frac{\partial(\eta G)}{\partial\eta}+H\frac{\partial G}{\partial\zeta}+FG=\frac{\partial^2 G}{\partial\zeta^2}$$

$$\eta F\frac{\partial Q}{\partial\eta}+H\frac{\partial Q}{\partial\zeta}=\frac{1}{\sigma}\frac{\partial^2 Q}{\partial\zeta^2}+\eta^2\left[\left(\frac{\partial F}{\partial\zeta}\right)^2+\left(\frac{\partial G}{\partial\zeta}\right)^2\right] \qquad (3.3)$$

Boundary conditions (2.10) are now replaced by the following conditions:

$$\begin{array}{lll} Q(\eta,0)=Q_w, & F(\eta,0)=0, & G(\eta,0)=1, \\ Q(\eta,\infty)=1, & F(\eta,\infty)=0, & G(\eta,\infty)=0, \end{array} \quad H(\eta,0)=0 \qquad (3.4)$$

Function D does not enter the system of equations (3.3) and can be found from Equations (2.7) and (2.9)

$$D=1/Q \qquad (3.5)$$

Close perusal of system (3.3) reveals, firstly, that functions $F(\eta,\zeta)$, $G(\eta,\zeta)$ and $H(\eta,\zeta)$ can be found independently of the form of the function $Q(\eta,\zeta)$ from the first three equations of the system and that the fourth equation allows us to find $Q(\eta,\zeta)$; secondly, that if we assume functions F, G, and H to be independent of η, the above mentioned first three differential equations of the system become ordinary ones and assume exactly the same form as the corresponding ones in the Karman incompressible fluid problem

$$2F+H'=0,\qquad F''-HF'-F^2=-G^2,\qquad G''-HG'-2FG=0 \qquad (3.6)$$

with boundary conditions

$$F(0)=0,\quad F(\infty)=0,\quad G(0)=1,\quad G(\infty)=0,\quad H(0)=0 \qquad (3.7)$$

Results of numerical solution of this system are given in many treatises and textbooks (for instance, in [5]).

The last equation of system (3.3) can be solved with the help of a rule suggested by Millsaps and Pohlhausen [2]. This is not a rule which is universally applicable, but it can be used for our case of constant disc temperature, and also for several varieties of temperature boundary conditions. Following Millsaps and Pohlhausen we represent function Q by three terms

$$Q(\eta,\zeta)=(Q_w-1)\,Q_1(\zeta)+\eta^2 S(\zeta)+1 \qquad (3.8)$$

After this the given equation resolves itself into two ordinary differential equations with corresponding boundary conditions

$$Q_1''-\sigma HQ_1'=0,\qquad Q_1(0)=1,\qquad Q_1(\infty)=0 \qquad (3.9)$$

$$S''-\sigma HS'+\sigma H'S=-\sigma(F'^2+G'^2),\qquad S(0)=0,\qquad S(\infty)=0 \qquad (3.10)$$

Solutions of both equations are known from the cited references [2]. We have thus arrived at an exact solution for the boundary layer equation of a rotating disc with boundary conditions (2.10) for $n = 1$ and for a finite value of Prandtl number σ.

4. Some numerical results. We can make use of the solution we have built up to afford a concrete evaluation of the effect of friction and heat transfer on the flow in the boundary layer of the disc. Let us do this for gas with Prandtl number $\sigma = 0.72$.

The frictional shear stress on the disc surface in dimensional form is

$$\tau_{z\theta} = \left(\mu \frac{\partial u_\theta}{\partial z}\right)_w = \rho_\infty \sqrt{\nu_\infty \omega^3} r G'(0) \tag{4.1}$$

The coefficient of frictional torque on one side of a disc of radius r_0 is

$$C_M = -\frac{2}{\rho_\infty{}^2 r_0{}^3 \omega^2 \pi r_0{}^2} \int_0^{r_0} 2\pi r^2 \tau_{z\theta}\, dr = -\frac{G'(0)}{\sqrt{R_\infty(r_0)}} = \frac{0.616}{\sqrt{R_\infty(r_0)}} \tag{4.2}$$

In a similar manner the heat flux due to conductivity, in the direction of the disc, per unit time and unit disc surface, can be expressed thus:

$$q_z = \left(\lambda \frac{\partial T}{\partial z}\right)_w = \lambda_\infty (T_w - T_\infty) \sqrt{\frac{\omega}{\nu_\infty}} \left[Q_1'(0) + \frac{\omega^2 r^2}{c_p (T_w - T_\infty)} S'(0) \right] \tag{4.3}$$

We then have a dimensionless coefficient of heat transfer for one side of a disc

$$C_E = -\frac{r_0}{\pi r_0{}^2 \lambda_\infty |T_w - T_\infty|} \int_0^{r_0} 2\pi r q_z\, dr =$$

$$= \operatorname{sign}(T_w - T_\infty) r_0 \sqrt{\frac{\omega}{\nu_\infty}} \left[Q_1'(0) + \frac{\omega^2 r_0{}^2}{2C_p T_\infty} \frac{T_\infty}{T_w - T_\infty} S'(0) \right] =$$

$$= \operatorname{sign}(T_w - T_\infty)\, r_0 \sqrt{\frac{\omega}{\nu_\infty}} \left(0.329 - 0.108 \frac{\omega^2 r_0{}^2}{C_p T_\infty} \frac{T_\infty}{T_w - T_\infty} \right) \tag{4.4}$$

Formulas (4.1) and (4.2) representing the effect of friction are analogous in structure to the corresponding formulas for the case of incompressible fluid; whilst, with regard to the influence of heat transfer, determined by formulas (4.3) and (4.4), it is expressed here in a slightly different way than with an incompressible fluid. The first difference consists in that the normal Prandtl number σ serves as the parameter on which functions Q_1 and S depend, and not the modified one as in Millsap's and Pohlhausen's problem. The second difference is the absence in (4.3) and (4.4) of terms containing derivatives of function Q introduced by these authors.

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